Frequency Detection from Multiplexed Compressed Sensing of Noisy Signals

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***Abstract*—Signal acquisition under a compressed sensing scheme offers the possibility of acquisition and reconstruction of signals sparse on some basis incoherent with measurement kernel with sub-Nyquist number of measurements. In particular when the sole objective of the acquisition is the detection of the frequency of a signal rather than exact reconstruction, then an undersampling framework like CS is able to perform the task. In this paper we explore the possibility of acquisition and detection of frequency of multiple analog signals, heavily corrupted with additive white Gaussian noise. We improvise upon the MOSAICS architecture proposed by us in our previous work to include a wider class of signals having non-integral frequency components. This makes it possible to perform multiplexed compressed sensing for general frequency sparse signals.**

Keywords—Multiplexed signal acquisition, Sub-Nyquist sampling, Non-integer frequency, A/D Conversion, MUSIC, Spectral Compressed Sensing, Signal reconstruction

I. INTRODUCTION

Detection of the frequency of single tone signals is the sole objective in some biomedical, communications and geological applications. Once the frequency of the signal is known it can be easily reconstructed. Identification of a single frequency component, even in the presence of heavy noise, can be nicely addressed as a sparse recovery problem. Consequently, a host of under sampling schemes can be employed which offer the advantage of reduced number of analog to digital converters (ADCs) in the system design, thereby, reducing cost, size and power consumption. In particular compressed sensing (CS) has provided solutions to many applications requiring reconstruction of signals which are sparse when transformed to a suitable basis. The focus in most of the work in CS literature has been towards reconstructing a signal after taking sub-Nyquist number of measurements. Simultaneous real time acquisition of multiple analog channels in an under sampling scheme has not been addressed at length. In our previous work, MOSAICS: Multiplexed Optimal Signal Acquisition Involving Compressed Sensing [4], we have proposed an architecture in which the idle Nyquist sample times in a CS based analog-to-information block are better utilized to sample multiple sparse signals in a multiplexed fashion. The net advantage is in terms of reduced number of ADCs in a complete data acquisition system for capturing multiple sparse signals. The MOSAICS scheme is independent of the CS recovery algorithm as it provides the architecture for simultaneous multiplexed acquisition of multiple channels each of which could possibly employ its own reconstruction algorithm. In the work presented in this paper, we have shown that MOSAICS can be effectively applied to the problem of detecting the frequency component in a signal, which being a comparatively easier problem, can be addressed even under low SNR conditions. A limitation in MOSAICS was the ability to handle signals whose frequency components are integer multiples of the fundamental frequency of DFT, since only in such a case are the DFT coefficients sparse. In practice for most real world signals having components whose frequencies are non-integral multiples of the fundamental, owing to spectral leakage due to sinc convolution, the sparsity constraint for a CS recovery algorithm is violated. In this work we show empirical evidence that mere detection of non-integral frequency components is a simpler problem that can be handled by MOSAICS even in the presence of heavy noise. A brief description of MOSAICS is given in Section II to create a platform for subsequent sections. The limitation of MOSAICS with respect to the acquisition of signals with non-integral frequency components is presented in Section III. In section III we also present the central idea behind the Spectral Compressed Sensing (SCS) method [5] which provides a remedy for the limitation. MOSAICS can easily incorporate SCS techniques for detecting the signal frequency more accurately as demonstrated in Section IV.

II. BACKGROUND

A. Input signal class

Consider a class Λ of continuous time (CT) real signals such that a signal $\mathbf{X} \in \Lambda$, if $\mathbf{X}$ as a function of time is composed of segments $\mathbf{x}_k$, such that

$$\mathbf{X}(t) = \mathbf{x_k}(t) \text{ for } t_{k-1} \leq t \leq t_k \text{ and } k = 1,2,\ldots,\infty. \quad (1)$$

Let R_{BW} be the bandwidth of the signal. Each $\mathbf{x_k}$, henceforth called as 'Locally Fourier Sparse' (LFS) segment, has a single frequency component with frequency less than R_{BW}. $R_{NYQ} = 2R_{BW}$ is the Nyquist sampling frequency. Consequently, since the signal is real, within any LFS segment the DFT has precisely two significant components and is therefore sparse. Further, let $\Delta = \inf\{l_k \in \mathbb{R} : l_k = t_k - t_{k-1}, \forall k = 1,2,\dots,\infty\}$ be the minimum duration of any LFS segment. In other words, the frequency of the signal does not change during this interval. Δ for a particular class of signals can be specified a priori.

B. *Recovery of signal*

A vector $\boldsymbol{\xi} \in \mathbb{R}^{\eta\times 1}$obtained by sampling any LFS segment $\mathbf{x}_k$ in the time interval $[\tau_1, \tau_2]$, where $t_{k-1} \le \tau_1$, $\tau_2 \le t_k$, at the Nyquist time instants τ_1, $\tau_1 + t_{NYQ}$, $\tau_1 + 2t_{NYQ}$, … $\tau_1 + (\eta - 1)t_{NYQ}$ where $t_{NYQ} = 1/R_{NYQ}$ is called a reconstruction segment (RS). One or more RS can be sequentially reconstructed to recover an LFS segment. The central tenet of a CS based acquisition is that even with measurements at only $\theta < \eta$, Nyquist time instants in the RS, it is possible to recover the signal using one of many CS recovery algorithms. Reduction in number of measurements has a direct consequence on the number of A/D conversions. The measurement vector obtained after undersampling, $\mathbf{f} \in \mathbb{R}^{\theta\times 1}$ is given by

$$\mathbf{f} = \boldsymbol{\phi}\boldsymbol{\xi} \tag{2}$$

Here $\boldsymbol{\phi} \in \mathbb{R}^{\theta\times\eta}$ is known as the measurement matrix formed by picking up θ randomly chosen rows of the identity matrix $\mathbf{I} \in \mathbb{R}^{\eta\times\eta}$.

Let the matrix, $\mathbf{F^{-1}} \in \mathbb{C}^{\eta\times\eta}$ represent the inverse discrete Fourier transform (IDFT) matrix. The column vector $\mathbf{g} \in \mathbb{C}^{\eta\times 1}$ of DFT coefficients corresponding to the time domain vector $\boldsymbol{\xi}$, satisfies

$$\boldsymbol{\phi}\mathbf{F^{-1}g} = \mathbf{f} \tag{3}$$

Solution of (3) can be substituted in $\boldsymbol{\xi} = \mathbf{F^{-1}g}$ to recover the reconstruction segment $\boldsymbol{\xi}$. However, the problem lies in the fact that the number θ of equations in the set (3) is lesser than the number η of unknowns. This implies that the solution to (3) is not unique. However, owing to the fact that $\boldsymbol{\xi}$ is LFS with only a single frequency component, the vector $\mathbf{g}$ is sparse and under a CS [1]-[3] framework, a solution to the undetermined set of equations, when treated as a convex optimization problem, can be obtained as

$$\mathbf{g} = \arg\min\|\mathbf{h}\|_1 \text{subject to } \boldsymbol{\phi}\mathbf{F^{-1}h} = \mathbf{f} \tag{4}$$

$$\text{followed by } \hat{\boldsymbol{\xi}} = \mathbf{F^{-1}g} \tag{5}$$

If ADC measurement noise needs to be considered, the constraint in (4) can be altered as,

$$\mathrm{M}(\boldsymbol{\phi}\mathbf{F^{-1}h} - \mathbf{f}) < \epsilon \tag{6}$$

where, M is defined as, $\mathrm{M}(\mathbf{v}) = \|\mathbf{v}\|_2 / \sqrt{N} \quad \forall v \in \mathbb{R}^{N\times 1}$ and ϵ is the maximum rms noise given as

$$\epsilon = \frac{\mathrm{M}(\mathbf{f})}{10^{\left(\frac{SMNR}{20}\right)}} \tag{7}$$

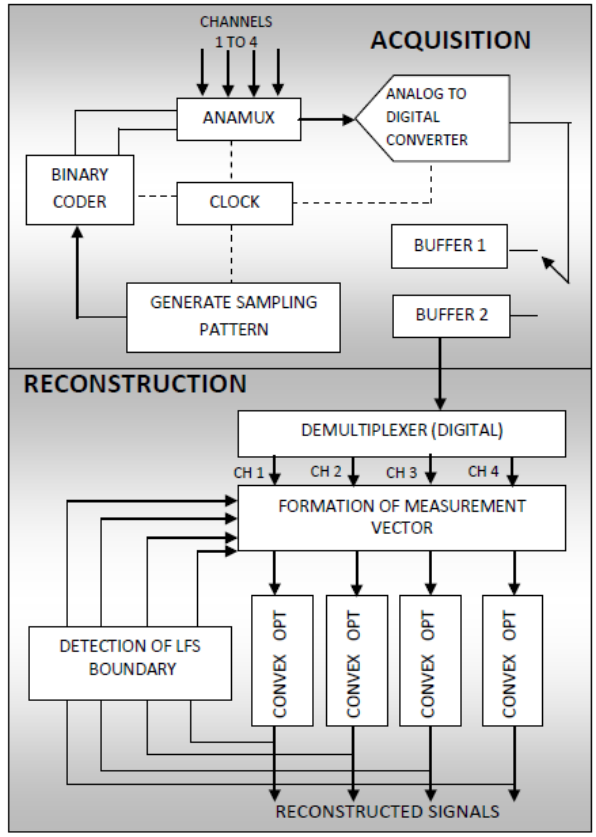

Fig. 1 Multiplexed acquisition under MOSAICS architecture

where, $SMNR = 20\log\left(\frac{\mathrm{M(f)}}{\epsilon}\right)$, is the signal-to-measurement noise ratio in dB.

C. *Detection of LFS boundary*

In order to minimize reconstruction error, consecutive RSs overlap in MOSAICS. The initial and significant portion of the measurement vector $\mathbf{f}$, comprises of samples which have already been reconstructed previously. Towards the end it has samples obtained by direct measurement on the signal at randomly spaced time instants. By virtue of overlap with previously reconstructed values, even if the reconstruction window slides over an LFS boundary, the length of the signal where there is a reconstruction error is minimized. A discrete jump in the DFT coefficients, quantitatively observed as a significant $'l2\ norm'$distance between the successive '$\mathbf{g}$' vectors indicates the occurrence of an LFS boundary.The RS constructed immediately after a boundary is detected does not use the overlapped samples from the previous RS. Overlap will continue for subsequent RSs until another boundary is detected.

D. *Simultaneous acquisition of multiple signals*

The fundamental idea of MOSAICS is with respect to making use of all the η time instants of the ADC. Under a classical CS acquisition scheme, only θ time instants of the ADC out of the possible η Nyquist time instants are used to acquire a single channel. In MOSAICS, the remaining $\eta - \theta$ Nyquist sampling instants are used to sample more than one analog signal. Consider for example, the acquisition of bandlimited analog signals with corresponding bandwidths R_{BW}^{i}, $i = 1 \ldots \mathcal{N}$. Each signal comprises contiguous LFS segments of minimum length L_i, each having exactly one frequency component. MOSAICS (shown in Figure 1) consists of two blocks – the Acquisition Block and the Reconstruction Block. The signals are acquired in a sequence of acquisition cycles. The Acquisition block operates at a frequency,

$$R_{OP} = 2\left(\max\left\{R_{BW}^{i}, i = 1..\mathcal{N}\right\}\right) \quad (8)$$

The sampling rate of the ADC in MOSAICS is R_{OP}. As soon as new acquisition cycle starts, the sampling pattern generator, releases a sequence of β numbers corresponding to the channel numbers $i = 1 \ldots \mathcal{N}$. The channel numbers are all interspersed like in a mosaic. At each sampling clock (at the rate of R_{OP}), the sampling pattern generator releases the next element in the sequence which is fed to the analog multiplexer after being converted to binary by the coder. The multiplexer selects the corresponding channel which the ADC has to sample. The sampled data is put in the active buffer. Once the acquisition cycle is completed, the active buffer is full and the reconstruction block reads the contents of the buffer. The interspersed samples are separated into individual channels inside the digital demultiplexer which has a copy of the sampling pattern. As soon as the number of samples required for a reconstruction window for any channel is available, the signal is reconstructed by convex optimization. Detection of an LFS boundary, if any, is conveyed to the algorithm. While the reconstruction block is operating, the acquisition block writes fresh data into the other buffer.

III. SIGNALS WITH NON-INTEGRAL FREQUENCIES

A. Limitation of MOSAICS

Let the frequency of the signal being acquired be f_0, the sampling frequency of MOSAICS be f_M and the length of the reconstruction window chosen under MOSAICS be η. The number of cycles of the signal in the reconstruction window is given by $N = \eta \frac{f_0}{f_M}$. If N is not an integer, then the set of DFT coefficients of the reconstruction window will not be sparse. We choose to call such frequencies like f_0, as non-integral frequencies. The reconstruction of the signal through l_1-minimization, cast as a convex optimization problem, fails. By arbitrarily increasing η in powers of 10, N can be made an integer. However, this does not provide a remedy to the problem due to two reasons. Firstly, if we take a reconstruction window larger than the minimum LFS segment length, then the RS is no longer DFT sparse. Secondly, as the size of the RS increases the computation becomes unwieldy.

B. Recovery methods

In the work reported in this paper, we employ a different reconstruction algorithm in MOSAICS, instead of the classical convex optimization for recovery of practically smooth signals with non-integral frequencies. This idea is presented in [5] under the name of "Spectral Compressed sensing". In this work the authors discuss at length various alternatives. If the signal representation is changed to a zero-padded DFT, the DTFT of the signal is more closely sampled and the DFT basis becomes a redundant frame of sinusoids known as a DFT frame. By increasing the zero-padding, and thus, the size of the frame, the signals with non-integral frequency components become more compressible, thereby, enhancing chances of recovery. This however leads to the frame becoming more coherent [6, 7] which is detrimental to recovery. Coherence of DFT frame is nothing but the spectral leakage problem which has been classically dealt with by tapering the signal with a suitable window function before applying the DFT [8, 9]. Techniques based on eigen analysis of the signal's correlation matrix have provided better solutions. Such methods give the line spectrum of the signal [8]. These algorithms estimate the principal components in the signal's autocorrelation matrix which give the dominant signal modes in the frequency domain. They provide better resolution of the parameters of a frequency sparse signal.

C. Application of MUSIC algorithm

One such algorithm is MUSIC that, given the signal vector **x,** first calculates the eigen decomposition of the autocorrelation matrix R_{XX} of size $P \times P$. It then obtains the P eigen values, $\lambda_1, \lambda_2, \ldots \lambda_P$ in decreasing order of magnitude and the corresponding eigen vectors, $v_1, v_2, \ldots v_P$. A score function is evaluated which returns the function peaks corresponding to the K largest scores. These correspond to the frequencies present in the signal. The line spectrum is thus found out as a K-sparse approximation algorithm in the frequency domain. From the K prominent frequencies, we estimate the values of the corresponding DTFT coefficients.

D. Greedy approach based on MUSIC

The reconstruction if a compressively sensed signal makes use of an iterative algorithm that picks up the K frequencies with the help of MUSIC. This is done like in greedy compressed sensing methods like Orthogonal Matching Pursuit [10]. The measurement vector becomes the initial residue. The residue in each of the succeeding iterations is obtained as

$$\mathbf{res} = \mathbf{f} - \boldsymbol{\phi}\boldsymbol{\xi}_{\mathbf{i-1}} \quad (9)$$

where **f** is the measurement vector, $\boldsymbol{\phi}$ is the measurement matrix, $\boldsymbol{\xi}_{i-1}$ is the reconstruction of the preceding iteration $i-1$. The new estimate is obtained from the residue,

$$\hat{\boldsymbol{\xi}}_i = \boldsymbol{\xi}_{i-1} + \boldsymbol{\phi}^T \times \mathbf{res} \quad (10)$$

The input to the MUSIC algorithm is the new estimate and K, the desired number of frequencies. The output of the algorithm is the K prominent frequencies $\widehat{\omega}_k$ together with the corresponding coefficients, $\hat{a}_k$. The reconstruction for the current iteration, $\boldsymbol{\xi}_i$ is obtained as the linear combination of the K chosen sinusoids.

$$\boldsymbol{\xi}_i = \sum_{k=1}^{K} \hat{a}_k \mathbf{e}(\widehat{\omega}_k) \quad (11)$$

E. MOSAICS with MUSIC

The MOSAICS architecture is modified by reconstructing the signal with the greedy MUSIC algorithm instead of

TABLE 1
FREQUENCY CHARACTERISTIC OF FOUR DIFFERENT SIGNALS

CHANNEL 1									CHANNEL 2								
Time in sec	Freq. in Hz	Detected frequencies at various SNR in dB							Time in sec	Freq. in Hz	Detected frequencies at various SNR in dB						
		-5	0	5	10	15	20				-5	0	5	10	15	20	
0 - 8	4.2	4.16	4.18	4.2	4.19	4.2	4.2		0 – 7	5.6	5.75	5.57	5.6	5.6	5.6	5.6	
8 - 17	8.7	8.69	8.69	8.7	8.7	8.7	8.7		7–14.8	11.9	11.5	11.2	11.2	11.1	11.2	11.2	
17 - 28	6.3	10.2	6.3	6.3	6.3	6.3	6.3		14.8 – 23	7.9	7.9	7.87	7.9	7.9	7.9	7.9	
28 - 35	16.9	16.5	16.8	16.9	16.9	16.9	16.9		23 –31	17.5	17.5	17.5	17.5	17.5	17.5	17.5	
>35	13.4	13.4	13.4	14.1	13.41	13.4	13.4		>31	14.9	18.3	14.9	14.9	14.9	14.9	14.9	

CHANNEL 3								CHANNEL 4							
Time in sec	Freq. in Hz	Detected frequencies at various SNR in dB						Time in sec	Freq. in Hz	Detected frequencies at various SNR in dB					
		-5	0	5	10	15	20			-5	0	5	10	15	20
0–11.4	10.5	11.4	10.6	10.5	10.5	10.5	10.5	0 - 9.5	8.5	16.8	8.5	8.5	8.5	8.5	8.5
11.4 -19	4.4	4.3	4.45	4.4	4.39	4.4	4.4	9.5 – 16.7	11.2	11.1	11.1	11.2	11.2	11.2	11.2
19–25	9.1	7.05	17.9	8.4	8.1	9.0	9.0	16.7- 29	18.4	15.2	19.5	19.3	18.35	18.4	18.4
25–32.6	15.6	15.7	15.5	15.5	12.4	15.6	15.6	29 – 37.5	15.6	9.32	15.4	17.4	15.2	14.8	16

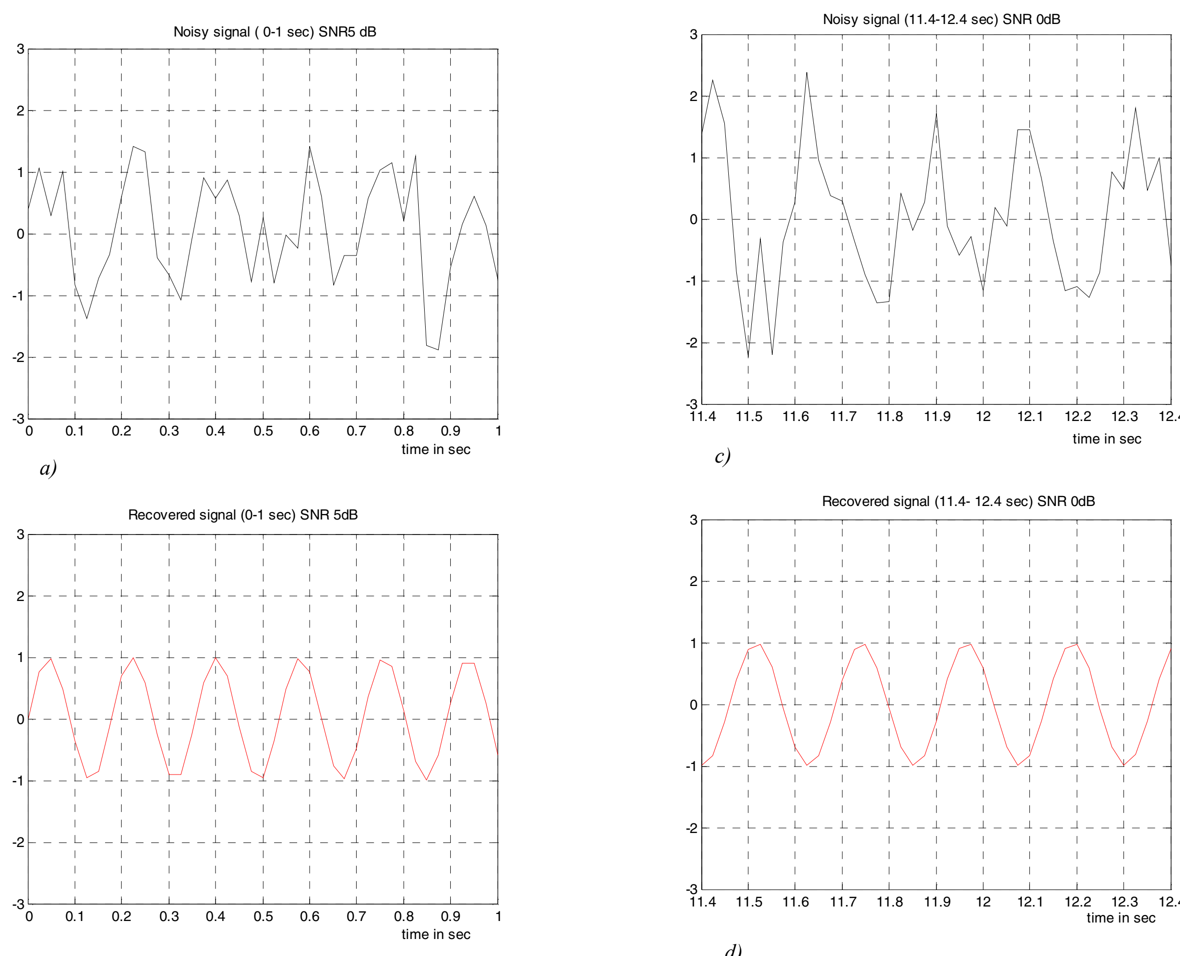

Figure 2 Reconstruction of noisy signals (a) and (b) Noisy signal (SNR 5dB)with a single tone of 5.6 Hz sampled in Channel 2 during the interval 0-1 sec and corresponding reconstructed signal. (c) and (d) Noisy signal (SNR 0dB)with a single tone of 4.4 Hz sampled in Channel 3 during the interval 11.4 -12.4 sec and corresponding reconstructed signal

performing a convex optimization. In addition, an LFS boundary is detected by calculating the deviation between the measured signal samples falling in the non-overlapping portion of the RS and the signal values estimated from (11) at the corresponding sample locations. This is in contrast to the calculation of the deviation in the '**g**' vectors that was originally used in MOSAICS. Also, since we only consider single tone signals although in the presence of heavy noise,the MUSIC algorithm seeks the highest frequency component instead of K values. Once the frequency of the noisy signal is known, it can be synthesized as a regular sinusoid.

IV. SIMULATIONS AND RESULTS

As in [4], four CT signals are sampled under the modified MOSAICS scheme for a duration of 40 sec. Table 1 gives the time-frequency characteristics of four channels. The MOSAICS operating frequency is $R_{OP} = 40$ kHz. Additive white Gaussian noise at various SNRs from -5dB to 20 dB is added to signal in the buffer at the end of every acquisition cycle to simulate measurement noise using the Matlab function *awgn*. The table shows the detected frequency for each channel at each SNR in the corresponding column. Figure 2a shows a snapshot of the measured samples .in Channel 2, during the first 1 second, at an SNR of 5 dB. The signal is very noisy and the single frequency of 5.6 Hz is hardly visible from the snapshot. The MUSIC based reconstruction correctly estimates the frequency as seen in the table under the 5dB column. Once the frequency is known the signal can be synthesized as shown in Figure 2b. Thus from a very noisy measurement the signal has been recovered with almost zero error. It is worth reiterating here that the advantage in the modified MOSAICS scheme presented in this paper is that we only detect the frequency in the signal instead of reconstructing the signal. It is obvious that the overlapping of RSs in MOSAICS requires that the signal also be reconstructed in addition to the detection of frequency. In the presence of noise, the overlapped portion of the signal will be highly inaccurate, nevertheless the highest frequency component that is detected by MUSIC remains unaltered even with the inaccurate reconstruction. The table clearly shows that the frequency is detected very accurately and except in a few cases, particularly at very low SNRs (-5dB and 0 dB), the detection is accurate.

V. CONCLUSIONS

It has been clearly demonstrated that MOSAICS can be effectively used to recover noisy sinusoids even at very low SNR values. Since the primary objective in such an application is only the detection of the frequency, detection and reconstruction of the signal can be done with reasonable accuracy even in the presence of heavy noise. MOSAICS offers the advantage of requiring lesser number of analog to digital converters for multiple analog channels acquired in real time. Further, with the introduction of MUSIC based recovery, the architecture can be used for a wider class of signals including those whose frequency is a non-integer multiple of the DFT fundamental frequency as long as they are locally Fourier sparse. Although the test case considered consists of signals with frequencies in the range of a few Hertz, MOSAICS can be extended to signals at any frequency range as long as all the signals acquired are in the same band. At higher frequencies, only the parameters of MOSAICS will be changed. Further, the MOSAICS scheme is scalable to cater to a larger number of signals, since multiple MOSAICS blocks can be used, all of which share a common reconstruction block implemented in the hardware.

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